

General Static Axially-symmetric Solutions of (2+1)-dimensional Einstein-Maxwell-Dilaton Theory

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Abstract

We obtain the general static solutions of the axially symmetric (2+1)-dimensional Einstein-Maxwell-Dilaton theory by dimensionally reducing it to a 2-dimensional dilaton gravity theory. The solutions consist of the magnetically charged sector and the electrically charged sector. We illuminate the relationship between the two sectors by pointing out the transformations between them.

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I. INTRODUCTION

The low dimensional analogs of the 4-dimensional general relativity are useful models to extract the analytic information about the physics of gravitation, due to their vastly simplified dynamical content. Among particularly important models of this kind are the (2+1)-dimensional general relativity [1] [2] [3] and the various 2-dimensional dilaton gravity models [4], such as the Callan-Giddings-Harvey-Strominger theory [5] and the Jackiw-Teitelboim [6] theory. If we only consider the axially symmetric sector of the (2+1)-dimensional general relativity, the dynamics of the problem becomes essentially 2-dimensional. The focus of this paper is to obtain the general static solutions of the axially symmetric (2+1)-dimensional Einstein-Maxwell-Dilaton theory by dimensionally reducing it to a 2-dimensional dilaton gravity theory.

The dimensional reduction we use in this paper has been originally utilized in [7] to obtain general static spherically symmetric solutions of the D -dimensional ($D > 3$) Einstein-Maxwell-Scalar theories. However, as reported in that paper, the determination of the analytic solutions of the 3-dimensional theory requires a different method, so it was not discussed there. In this paper, we find it is actually rather straightforward to give an analytic treatment of the 3-dimensional case. In that sense this paper is a natural supplement for [7]. However, more important motivation for this work comes from the fact that the (2+1)-dimensional gravity itself is interesting. What we get in this paper is general static and axially-symmetric solutions. Thus, we recover as special cases the results of [2] where the magnetically charged, static, and axially symmetric solution is obtained, and [3], where one finds the electrically charged solution. Moreover we also include a dilaton field, which plays an important role in string theory, or other Klein-Kaluza type theories [8], in our general consideration.

In the following section, we present the dimensional reduction of the (2+1)-dimensional gravity theory to a 2-dimensional dilaton gravity theory. The geometrical property of the axially-symmetric (2+1)-dimensional space-time is different from that of the spherically

symmetric D -dimensional space-time. Thus, the treatment of $U(1)$ gauge field is quite different from the D -dimensional Einstein-Maxwell-Scalar theories. We are led to separately consider the electrically charged case and the magnetically charged case. The general static solutions of the resulting 2-dimensional dilaton gravity theory for electrically charged case are presented in Appendix, from which we can also get magnetically charged solutions by utilizing a electric/magnetic-duality-like transformation. Our results in Appendix are in itself interesting, since it gives exact general static solutions for a class of the 2-dimensional dilaton gravity theories not considered elsewhere in literature. The class of theories in Appendix contains, as its special case, the (2+1)-dimensional gravity which is the main concern of this paper.

II. GENERAL SOLUTIONS OF THE (2+1)-DIMENSIONAL GENERAL RELATIVITY

We consider the axially-symmetric reduction of the (2+1)-dimensional Einstein-Maxwell-Dilaton theory

$$I = \int d^3x \sqrt{g^{(3)}} (R^{(3)} - \frac{1}{2} g^{(3)ij} \partial_i f \partial_j f + \frac{1}{4} e^{\chi f} F^2) \quad (1)$$

to a 2-dimensional dilaton gravity theory. Here $R^{(3)}$ and $g_{ij}^{(3)}$ represent the (2+1)-dimensional scalar curvature and metric tensor, respectively, and Latin indices i, j run over the (2+1) space-time coordinate labels. We also have F the curvature 2-form for a $U(1)$ gauge field and f a dilaton field. We can write $F_{ij} = \partial_i A_j - \partial_j A_i$ in terms of the vector potential A_i . The non-zero value of the real parameter χ couples the dilaton f to the $U(1)$ gauge field in the manner found in the Klein-Kaluza theory or in the low energy target space effective theory of the string theory [8]. As the first step of the dimensional reduction, we write the axially symmetric (2+1)-dimensional metric as the sum of the longitudinal part (with 2-dimensional metric $g_{\alpha\beta}$ where Greek indices α, β run over the (1+1) space-time coordinate labels) and the transversal angular part

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta - e^{-4\phi}d\theta^2. \quad (2)$$

The transversal angular part corresponds to a unit circle in case of the axial symmetry, where θ corresponds to the angle of a point on the circle. For definiteness, we choose to describe the longitudinal 2-dimensional space-time in terms of conformal gauge with conformal coordinate x^\pm . In other words, we have $g_{\alpha\beta}dx^\alpha dx^\beta = -\exp(2\rho)dx^+dx^-$ for the longitudinal metric, where $\exp(2\rho)$ is the conformal factor. The ϕ field, the scale factor of the transversal metric, will be interpreted as the 2-dimensional dilaton field under the dimensional reduction. We use the $(+ - -)$ signature throughout this paper. The axial symmetry requires that the metric $g_{\alpha\beta}$, the dilaton field f and the 2-dimensional dilaton field ϕ do not depend on θ . The three components of the (2+1)-dimensional 2-form curvature, F_{-+} , $F_{+\theta}$, and $F_{-\theta}$ should not depend on θ either (in fact their vector potential can also be chosen to be independent of θ). However, unlike the higher dimensional spherical symmetry case, $F_{\pm\theta}$ do not have to vanish in general under the requirement of the axial symmetry.

The equations of motion from our action (1) are given by

$$R_{ij}^{(3)} - \frac{1}{2}R^{(3)}g_{ij}^{(3)} = T_{ij}^m \quad (3)$$

by varying the action with respect to the (2+1)-dimensional metric tensor,

$$D_i \left(e^{\chi f} F^{ij} \right) = 0 \quad (4)$$

for the $U(1)$ gauge field, and

$$g^{(3)ij}D_i D_j f + \frac{\chi}{4}e^{\chi f}F^2 = 0 \quad (5)$$

for the dilaton field f . Here D denotes the covariant derivative in (2+1)-dimensional space-time and T_{ij}^m is the stress-energy tensor of the $U(1)$ gauge field and the dilaton field f . After the imposition of the axial symmetry, all of the above equations, except the (\pm, θ) components of Eq. (3), can be obtained from a 2-dimensional dilaton gravity action

$$I = \int d^2x \sqrt{-g} e^{-2\phi} \left[R - \frac{1}{2}g^{\alpha\beta} \partial_\alpha f \partial_\beta f + \frac{1}{4}e^{\chi f} g^{\alpha\beta} g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} - \frac{1}{2}e^{\chi f + 4\phi} g^{\alpha\beta} F_{\alpha\theta} F_{\beta\theta} \right], \quad (6)$$

which is obtained from (1) by imposing the axial symmetry and integrating out the θ coordinate. We note that the sum over the repeated indices run only through x^+ and x^- , the 2-dimensional longitudinal space-time. By varying (6) with respect to the 2-dimensional metric $g_{\alpha\beta}$, we recover $(+, +)$, $(-, -)$ and $(-, +)$ component equations of (3). The (θ, θ) component of (3) is obtained by varying (6) with respect to the 2-dimensional dilaton field ϕ . The 2-form curvature of the gauge field F is composed of the electric field F_{-+} and the magnetic field $F_{\pm\theta}$. When we consider static and axially symmetric equations of motion, we can show that $F_{+\theta} = \partial_+ A_\theta = \partial_- A_\theta = F_{-\theta}$ since A_\pm does not depend on θ and, under a suitable choice of conformal coordinates, A_θ depends only on a space-like coordinate $x = x^+ + x^-$. Thus, $F_{\pm\theta}$ contains only magnetic fields and no electric fields (i.e., $F_{+\theta} - F_{-\theta} = 0$), as far as the static analysis is concerned. By inserting (2) into the (\pm, θ) components of the Einstein tensor $R_{ij}^{(3)} - g_{ij}^{(3)} R^{(3)}/2$, we can verify that they identically vanish. Consequently, the (\pm, θ) components of Eq. (3) reduce to

$$T_{\pm\theta}^m = \pm e^{\chi f - 2\rho} F_{+-} F_{\theta\pm} + \frac{1}{2} \partial_\pm f \partial_\theta f = 0,$$

which becomes

$$F_{+-} F_{\theta\pm} = 0 \tag{7}$$

upon imposing the axial-symmetry. Thus, our original system reduces to a 2-dimensional dilaton gravity action (6), which is of the type we solve in Appendix. The only missing information in (6) is supplied by Eq.(7); it simply states that we have either electrically charged solutions or magnetically charged solutions, but no dyonic solutions which have both magnetic and electric charges.

First, we consider the purely electrically charged case, for which we set $F_{\pm\theta} = 0$ and $F_{-+} \neq 0$. Then, we immediately find (6) reduces to (A1) in Appendix with the assignment of $\gamma = \mu = \epsilon = 0$. Thus, we can follow the calculations in Appendix leading to the action

$$I_e = \int dx [\dot{\Omega} \dot{\rho} - \frac{1}{4} \Omega \dot{f}^2 + \frac{1}{4} e^{\chi f - 2\rho} \Omega \dot{A}^2]. \tag{8}$$

where we introduce $\Omega = \exp(-2\phi)$ and a space-like coordinate $x = x^+ + x^-$. All the functions depend only on x , and we also introduce $F_{-+} = \dot{A}$ with the overdot representing the differentiation with respect to x . Getting the general static solution in the conformal gauge is tantamount to solving the equations of motion derived from the action (8) under the gauge constraint

$$\ddot{\Omega} - 2\dot{\rho}\dot{\Omega} + \frac{1}{2}\Omega\dot{f}^2 = 0. \quad (9)$$

From Eq. (A22), Eq. (A24), Eq. (A25) and Eq. (A26), we get

$$2|Q|e^{2\rho} = e^{\chi f_1} e^{2sI(A)} \quad (10)$$

$$f(A) = \frac{1}{\chi} (2sI(A) - \ln |P(A)|) + f_1 \quad (11)$$

$$\Omega^2 = e^{-4\phi_0} |P(A)|^{-2/\chi^2} e^{(8s^2 - f_2)I(A)/(2s\chi^2)} \quad (12)$$

$$x - x_0 = \int \frac{\Omega(A)}{P(A)} dA \quad (13)$$

where Q , s , f_0 , c , f_1 , ϕ_0 and x_0 are constants of integration [9] and

$$P(A) = (2s - \chi f_0)A + \frac{\chi^2}{2}QA^2 + c.$$

Here $Q \neq 0$, $f_2 = -\chi^2 f_0^2 + 4s\chi f_0 + 2\chi^2 Qc$ and $I(A) = \int P(A)^{-1} dA$. From Eq. (A27), we can rewrite the (2+1)-dimensional metric in the geometric gauge as

$$ds^2 = \frac{P}{2Q} e^{\chi f} \left[dT^2 - \frac{1}{16P^2} \left(\frac{dA}{d\phi} \right)^2 dr^2 \right] - r^2 d\theta^2, \quad (14)$$

where $r = \Omega$ and $2T = x^+ - x^-$ is the natural time-like coordinate orthogonal to the space-like coordinate $x = x^+ + x^-$.

Now we consider the purely magnetically charged case where $F_{-+} = 0$ and $F_{\pm\theta} \neq 0$. To obtain the general static solutions, we once again assume all fields depend on a single space-like variable $x = x^+ + x^-$ under a suitable choice of the conformal gauge. The static magnetic field can, thus, be written as $F_{-\theta} = F_{+\theta} = \dot{A}$, where the overdot represents the differentiation with respect to x , as before. Then, the static equations of motion from (6) can be derived from the following action

$$I_m = \int dx [\dot{\Omega}\dot{\rho} - \frac{1}{4}\Omega\dot{f}^2 - \frac{1}{4}e^{\chi f}\Omega^{-1}\dot{A}^2] \quad (15)$$

along with the gauge constraint whose static version is given by the condition

$$\ddot{\Omega} - 2\dot{\rho}\dot{\Omega} + \frac{1}{2}\Omega\dot{f}^2 + \frac{1}{2}e^{\chi f}\Omega^{-1}\dot{A}^2 = 0. \quad (16)$$

We could follow the analysis in the fashion given in Appendix to solve the above equations of motion. However, we take an alternative route here. We observe that there exists a transformation of fields that maps (15) into (8). Namely, under the transformation \mathcal{T}_{me} defined by

$$\mathcal{T}_{me} : (\Omega, e^\rho, A, f) \rightarrow (e^\rho, \Omega, -iA, f), \quad dx \rightarrow e^\rho\Omega^{-1}dx, \quad (17)$$

the magnetic action (15) transforms exactly into the electric action (8). In addition to this, under the transformation \mathcal{T}_{em} defined as

$$\mathcal{T}_{em} : (\Omega, e^\rho, A, f) \rightarrow (e^\rho, \Omega, +iA, f), \quad dx \rightarrow e^\rho\Omega^{-1}dx, \quad (18)$$

the electric action (8) changes into the magnetic action (15). We note that both $\mathcal{T}_{em} \circ \mathcal{T}_{me}$ and $\mathcal{T}_{me} \circ \mathcal{T}_{em}$ are the identity map in the space of fields. In fact, recalling that $F_{+-} = \dot{A}$ in the electrically charged case and $F_{\pm\theta} = \dot{A}$ in the magnetically charged case, the above transformations are similar to the usual electric/magnetic duality transformations where one transforms $\vec{B} \rightarrow -\vec{E}$ (\mathcal{T}_{me} in our case) and $\vec{E} \rightarrow \vec{B}$ (\mathcal{T}_{em} in our case).

Given these transformations, it is straightforward to write down the solutions for the magnetic case utilizing our previous results for the electrically charged solutions. We introduce \mathcal{T}_{me} as a field redefinition

$$\Omega = e^{\bar{\rho}}, \quad e^\rho = \bar{\Omega}, \quad f = \bar{f}, \quad A = -i\bar{A}, \quad dx = e^{\bar{\rho}}\bar{\Omega}^{-1}d\bar{x}. \quad (19)$$

Then the equations of motion for the redefined fields become identical to those in the electrically charged case and, as a result, the integrated form of them are given in (A13)-(A16) in Appendix. We have

$$f_0 = \bar{\Omega}\bar{f}' + \frac{\chi}{2}e^{\chi\bar{f}-2\bar{\rho}}\bar{\Omega}\bar{A}'\bar{A} \quad (20)$$

$$2iQ = e^{\chi\bar{f}-2\bar{\rho}}\bar{\Omega}\bar{A}' \quad (21)$$

$$c_0 = \bar{\rho}'\bar{\Omega}' - \frac{1}{4}\bar{\Omega}\bar{f}'^2 + \frac{1}{4}e^{\chi\bar{f}-2\bar{\rho}}\bar{\Omega}\bar{A}'^2 \quad (22)$$

$$s + c_0\bar{x} = \bar{\rho}'\bar{\Omega}, \quad (23)$$

where the prime denotes the differentiation with respect to \bar{x} . The gauge constraint (16) changes into

$$\bar{\Omega}\bar{\rho}'' - \bar{\rho}'\bar{\Omega}' + \frac{1}{2}\bar{\Omega}\bar{f}'^2 - \frac{1}{2}e^{\chi\bar{f}-2\bar{\rho}}\bar{\Omega}\bar{A}'^2 = 0 \quad (24)$$

under the field redefinition. Using $\bar{\rho}''\bar{\Omega} = c_0 - \bar{\rho}'\bar{\Omega}'$ which is obtained by differentiating Eq. (23), we find that (24) precisely reduces to a condition $c_0 = 0$ just as (9) gives the same condition. Thus, we can write down the solutions (using original field variables) immediately as follows:

$$2Q\Omega^2e^{-\chi f} = (2s - \chi f_0)A - \frac{\chi^2}{2}QA^2 + c \equiv P(A) \quad (25)$$

$$f(A) = \frac{1}{\chi}(2sI(A) - \ln|P(A)|) + f_1 \quad (26)$$

$$e^{2\rho} = e^{-4\phi_0}|P(A)|^{-2/\chi^2}e^{(8s^2-f_2)I(A)/(2s\chi^2)} \quad (27)$$

$$x - x_0 = \int \frac{\Omega(A)}{P(A)}dA, \quad (28)$$

where $f_2 = -\chi^2 f_0^2 + 4s\chi f_0 - 2\chi^2 Qc$. These are the general static solutions for the magnetically charged case.

The transformations \mathcal{T}_{me} and \mathcal{T}_{em} are similar to the usual electric/magnetic duality. In general time-dependent (2+1)-dimensional case, there are two components for the electric field and a single component for the magnetic field. Thus, this is not a duality transformations in the usual sense of the 4-dimensional case. However, as far as static solutions are concerned, the number of components for both electric and magnetic fields is one, so the existence of duality-like transformations is not as bothering as it first seems.

If $F_{-+} = F_{\pm\theta} = 0$, the equations of motion are solved in Appendix. From Eq. (A17), Eq. (A18) and Eq. (A20), we have

$$f(z) = p \ln |\ln z| + p(\ln |2s| - \rho_0) \quad (29)$$

$$ds^2 = (\ln z)^2 dt^2 - |\ln z|^{p^2/2} c_1^{-2} z^{-2} (dz^2 + z^2 d\theta^2), \quad (30)$$

where $p = f_0/s$, $c_1^2 = |2s|^{-p^2/2} e^{\rho_0 p^2/2 - 2c_1/s}$, $\ln z = \pm e^\rho/(2s)$ and $dt = \pm 2s dT$. These results are the same as those given in [2].

In the absence of the dilaton field, namely if $f_0 = f_1 = \chi = 0$, our results reduce to the solutions found in the literature. For electrically charged solutions, Eq. (10) becomes

$$2Qe^{2\rho} = 2sA + c \quad (31)$$

and Eq. (12) yields [10]

$$\Omega = e^{-2\phi_0} e^{-QA/(2s)}. \quad (32)$$

The x -dependence is given from Eq. (13) as

$$x - x_0 = e^{-2\phi_0} \int \frac{e^{QA/(2s)}}{2sA + c} dA. \quad (33)$$

The (2+1)-dimensional metric becomes

$$ds^2 = 8Q^2 \ln\left(\frac{r_c}{r}\right) dt^2 - \frac{1}{8Q^2} \left(\ln\left(\frac{r_c}{r}\right)\right)^{-1} dr^2 - r^2 d\theta^2, \quad (34)$$

where $\ln r_c = Qc/(4s^2) - 2\phi_0$, $r = \Omega$ and $dt = \pm s/(2Q^2) dT$. The electric field is

$$F_{rt} = \mp \frac{Q}{r} \quad (35)$$

The above result is the same as the solution found by Gott *et al.* [3]. The magnetically charged solutions are given by carefully taking $\chi \rightarrow 0$ limit of Eqs. (26)-(28).

$$2Q\Omega^2 = 2sA + c \quad (36)$$

$$e^{2\rho} = e^{-4\phi_0} e^{QA/s} \quad (37)$$

$$\frac{dA}{dx} = 2Q\Omega \quad (38)$$

The (2+1)-dimensional metric for magnetic case is

$$ds^2 = \frac{e^{-4\phi_0 - cQ/(2s^2)}}{4s^2} e^{Q^2 r^2/s^2} [dt^2 - dr^2] - r^2 d\theta^2 \quad (39)$$

where $dt = \pm 2sdT$, as given in [2]. We note that even if the metric in (39) is related to the metric (34) via \mathcal{T}_{me} , the form of each metric looks quite different from each other in the geometric gauge. The choice of conformal gauge makes the existence of \mathcal{T}_{me} and \mathcal{T}_{em} clear.

III. DISCUSSIONS

We get the general axially symmetric static solutions of the (2+1)-dimensional Einstein-Maxwell-Dilaton theory in this paper. The reason for the difference between this case and the case of the D -dimensional Einstein-Maxwell-Scalar theories ($D > 3$) comes from the difference of the transversal space in each case. In (2+1)-dimensional axially symmetric geometry, we have an Abelian symmetry, while the rotational symmetries for $D > 3$ cases are non-Abelian. Thus, the s -wave sector of the (2+1)-dimensional theory contains the magnetic sector. Another indirect consequence of this difference is that the decoupled equation for the Ω field Eq. (A25) is the first order differential equation rather than the second order one in $D > 3$ case. This makes the analysis in this paper much simpler than that of [7]. This illustrates our general point that the low dimensional analogs of the 4-dimensional Einstein theory provide a more analytically tractable framework for the study of the gravitation.

The recent interest in (2+1)-dimensional gravity is partly due to [11] where one finds black hole solutions after adding the negative cosmological constant term to the gravity action we consider in this paper [12]. Thus, one of the most immediate generalizations of this paper is to add the negative cosmological constant to our action. It is interesting to note that the transformations \mathcal{T}_{em} and \mathcal{T}_{me} still exchange the magnetic and the electric sector of the theory even under this generalization, rather similar to the conventional electric/magnetic duality in the 4-dimensional Maxwell theory. Thus, in future attempts to solve the theory following our lines, it is sufficient enough to consider only the magnetic (or electric) sector of the theory. Additionally, it remains to be seen whether one can find dyonic solutions in

(2+1)-dimensional gravity coupled with a $U(1)$ gauge field, once we relax the condition of the rotational symmetry. The transformations \mathcal{T}_{em} and \mathcal{T}_{me} will be helpful in getting an answer to this question.

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APPENDIX A: SOLUTIONS OF A CLASS OF 2-DIMENSIONAL DILATON GRAVITY THEORIES

The action we consider here is given by

$$I = \int d^2x \sqrt{-g} e^{-2\phi} [R + \gamma g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \mu e^{2\lambda\phi} - \frac{1}{2} g^{\alpha\beta} \partial_\alpha f \partial_\beta f + \frac{1}{4} e^{\epsilon\phi + \chi f} F^2], \quad (\text{A1})$$

where R denotes the 2-dimensional scalar curvature and F the curvature 2-form for an Abelian gauge field. ϕ and f represent a dilaton field and a massless scalar field, respectively. The parameters γ , μ , λ , ϵ and χ are assumed to be arbitrary real parameters satisfying $4 - 2\lambda - \epsilon = 0$ and $4 - 2\lambda - \gamma + \epsilon = 0$. This case is not considered in [7] where they solved $4 - 2\lambda - \epsilon \neq 0$ and $4 - 2\lambda - \gamma + \epsilon = 0$ case.

We choose to work in a conformal gauge given by $g_{+-} = -e^{2\rho + \gamma\phi/2}/2$, $g_{--} = g_{++} = 0$, and choose the negative signature for a space-like coordinate and the positive signature for a time-like coordinate. In this conformal gauge our original action, modulo total derivative terms, is simplified to be

$$I = \int dx^+ dx^- (4\Omega \partial_+ \partial_- \rho + \frac{\mu}{2} e^{2\rho} \Omega^{-1} + \Omega \partial_+ f \partial_- f - e^{\chi f - 2\rho} \Omega F_{-+}^2), \quad (\text{A2})$$

where $\Omega = e^{-2\phi}$ and $F_{-+} = \partial_- A_+ - \partial_+ A_-$. The equations of motion in the conformal gauge are given by

$$\partial_+ \partial_- \Omega + \frac{\mu}{4} e^{2\rho} \Omega^{-1} + \frac{1}{2} e^{\chi f - 2\rho} \Omega F_{-+}^2 = 0, \quad (\text{A3})$$

$$\partial_+ \partial_- \rho - \frac{\mu}{8} \frac{e^{2\rho}}{\Omega^2} + \frac{1}{4} \Omega \partial_+ f \partial_- f - \frac{1}{4} e^{\chi f - 2\rho} F_{-+}^2 = 0, \quad (\text{A4})$$

along with the equations for the massless scalar field

$$\partial_+ \Omega \partial_- f + \partial_- \Omega \partial_+ f + 2\Omega \partial_+ \partial_- f + \chi e^{\chi f - 2\rho} \Omega F_{-+}^2 = 0, \quad (\text{A5})$$

and for the Abelian gauge fields

$$\partial_- (e^{\chi f - 2\rho} \Omega F_{-+}) = 0, \quad (\text{A6})$$

$$\partial_+ (e^{\chi f - 2\rho} \Omega F_{-+}) = 0. \quad (\text{A7})$$

The equations for the Abelian gauge fields can be solved to give

$$F_{-+} = e^{-\chi f + 2\rho} \Omega^{-1} Q, \quad (\text{A8})$$

where Q is a constant.

To get the solutions we need, in addition to the equations of motion, the gauge constraints resulting from the choice of the conformal gauge. They are given by

$$\frac{\delta I}{\delta g^{\pm\pm}} = 0, \quad (\text{A9})$$

where I is the original action Eq. (A1). We obtain the gauge constraints

$$\partial_\pm^2 \Omega - 2\partial_\pm \rho \partial_\pm \Omega + \frac{1}{2} \Omega (\partial_\pm f)^2 = 0. \quad (\text{A10})$$

Now we have to find the static solutions of the equations of motion Eq. (A3)-(A7) with the constraints Eq. (A10). The general static solutions can be found by assuming all functions except the gauge field depend on a single space-like coordinate $x = x^+ + x^-$. Then from Eq. (A8) we observe that the variable F_{-+} automatically becomes dependent only on x , and we can consistently reduce the partial differential equations into the coupled second order ordinary differential equations (ODE's). The resulting ODE's except the gauge constraint can be derived from an effective action

$$I = \int dx [\dot{\Omega}\dot{\rho} - \frac{\mu}{8}e^{2\rho}\Omega^{-1} - \frac{1}{4}\Omega\dot{f}^2 + \frac{1}{4}e^{\chi f - 2\rho}\Omega\dot{A}^2], \quad (\text{A11})$$

where the overdot represents taking a derivative with respect to x and $\dot{A} = F_{-+}$. The gauge constraints become

$$\ddot{\Omega} - 2\dot{\rho}\dot{\Omega} + \frac{1}{2}\Omega\dot{f}^2 = 0. \quad (\text{A12})$$

The general solutions of the above ODE's are the same as the general static solutions of the original action under a particular choice of the conformal coordinates.

The equations of motion can be integrated to the coupled nonlinear first order ODE by constructing Noether charges of the effective action. We observe the following four continuous symmetries of the action Eq. (A11):

$$f \rightarrow f + \alpha, A \rightarrow Ae^{-\chi\alpha/2}$$

$$A \rightarrow A + \alpha$$

$$x \rightarrow x + \alpha$$

$$x \rightarrow xe^\alpha, \Omega \rightarrow \Omega e^\alpha,$$

where α is an arbitrary real parameter of each transformation. The Noether charges for these symmetries are constructed as:

$$f_0 = \Omega\dot{f} + \frac{\chi}{2}e^{\chi f - 2\rho}\Omega\dot{A}A \quad (\text{A13})$$

$$2Q = e^{\chi f - 2\rho}\Omega\dot{A} \quad (\text{A14})$$

$$c_0 = \dot{\rho}\dot{\Omega} - \frac{1}{4}\Omega\dot{f}^2 + \frac{\mu}{8}e^{2\rho}\Omega^{-1} + \frac{1}{4}e^{\chi f - 2\rho}\Omega\dot{A}^2 \quad (\text{A15})$$

$$s + c_0x = \dot{\rho}\Omega. \quad (\text{A16})$$

Note that the third Noether charge c_0 is fixed to be zero ($c_0 = 0$) by using the gauge constraint Eq. (A12) and the equation of motion for ρ which is derived from the effective action Eq. (A11).

First, we find solutions when there is no $U(1)$ gauge field. In case of $s \neq 0$, using Eq. (A13), Eq. (A16) and Eq. (A15), we get solutions for f and Ω in terms of ρ as

$$f = \frac{f_0}{s}(\rho - \rho_0) \quad (\text{A17})$$

$$\Omega = e^{c_1/s} e^{f_0^2(\rho - \rho_0)/(4s^2)} \exp \left[\mu e^{2\rho}/(16s^2) \right], \quad (\text{A18})$$

where ρ_0 and c_1 are constants of integration. From Eq. (A16) we get x -dependence of ρ as

$$x - x_0 = s^{-1} \int \Omega(\rho) d\rho, \quad (\text{A19})$$

where x_0 is a constant of integration. The metric becomes

$$ds^2 = 4s^2(\ln z)^2 dT^2 - \frac{\Omega^2}{z^2}(dz^2 + z^2 d\theta^2) \quad (\text{A20})$$

where $\ln z = \pm e^\rho/(2s)$ and $2T = x^+ - x^-$. In case of $s = 0$ we get

$$\begin{aligned} \rho &= \rho_0 \\ f &= \int \frac{f_0}{\Omega(x)} dx + f_1 \\ \Omega(x) &= \text{arbitrary function,} \end{aligned}$$

where ρ_0 and f_1 are constants of integration and $2f_0^2 = \mu e^{2\rho_0}$. The metric becomes

$$ds^2 = -e^{2\rho_0} \left[\frac{1}{4} \left(\frac{d\Omega}{dx} \right)^2 dr^2 - dT^2 \right]. \quad (\text{A21})$$

where $r = \Omega$.

Second, we find solutions when the $U(1)$ gauge field does not vanish. From Eq. (A13), Eq. (A16) and Eq. (A14) we get

$$2Qe^{2\bar{\rho}} = (2s - \chi f_0)A + \frac{\chi^2}{2}QA^2 + c \equiv P(A), \quad (\text{A22})$$

where $\bar{\rho} = \rho - \chi f/2$ and c is a constant of integration. From Eq. (A13), (A14) and (A22) we can determine f via

$$\dot{f} = \frac{f_0 - \chi QA}{P(A)} \dot{A} \quad (\text{A23})$$

which upon integration becomes

$$f(A) = \frac{2s}{\chi} I(A) - \frac{1}{\chi} \ln |P(A)| + f_1, \quad (\text{A24})$$

where $I(A) = \int P(A)^{-1} dA$ and f_1 is a constant of integration. The constant of integration f_1 represents the trivial constant term which we can add to the scalar field f . Using Eq. (A14) and (A22), we can rewrite Eq. (A15) as

$$8s \frac{d\phi}{dA} = \frac{4sQA + 2Qc - f_0^2}{P(A)} + \frac{\mu}{4Q} e^{\chi f}. \quad (\text{A25})$$

By integrating the above equation we get $\Omega(A)$ as a function of A . Since Q does not vanish, we can find A as a function of x by plugging Eq. (A22) into Eq. (A14),

$$x - x_0 = \int \frac{\Omega(A)}{P(A)} dA, \quad (\text{A26})$$

where x_0 is the constant of integration. The metric is given by

$$ds^2 = -\frac{P}{2Q} e^{\chi f - \gamma \phi/2} \left[\frac{1}{16P^2} \left(\frac{dA}{d\phi} \right)^2 dr^2 - dT^2 \right]. \quad (\text{A27})$$

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